Indian Statistical Institute, Bangalore Centre

B.Math (Hons) II Year, Second Semester

Backpaper Examination

Optimization

Time: 3 Hours May , 2012 Instructor: Pl.Muthuramalingam

Maximum mark you can get is:50

For neatness 01 mark.

1. Let $g, f : R^2 \to R$ be c^1 functions. Let $\mathbf{x}$ be a local minima for f on $S = \{\mathbf{y} : g(\mathbf{y}) = c\}$ where c is a constant. State and prove Lagrange method. [5]

2. Let J be an interval and $f : J \to R$ any concave function. Let $x_1 < x_2 < x_3$. Show that the following inequality connecting the difference quotients hold.

$$\frac{f(x_3) - f(x_2)}{x_3 - x_2} \leq \frac{f(x_3) - f(x_1)}{x_3 - x_1} \leq \frac{f(x_2) - f(x_1)}{x_2 - x_1}.$$

[4]

3. Let $V = A \bigoplus B$ where A, B, V are all finite dimensional linear spaces. If $a_1, a_2, \cdots a_r \varepsilon A$ are linearly independent and $b_1, b_2, b_r \varepsilon B$, then $\nu_1, \nu_2, \cdots, \nu_r$ given by $\nu_j = (a_j, b_j)$ are linearly independent. [2]

4. Let

$$P = \left\{ \begin{array}{ll} (x_1, x_2) \in R^2 : & x_1 + x_2 \leq 40, \\ & 2x_1 + x_2 \leq 60 \\ & x_1 \leq 20 \\ & x_1, x_2 \geq 0 \end{array} \right\}.$$

a) Show that

$$P = \left\{ \begin{array}{ll} (x_1, x_2) \in R^2 : & x_1 + x_2 \leq 40 \\ & x_1 \leq 20 \\ & x_1, x_2 \geq 0 \end{array} \right\}.$$

[1]

b) Draw the figure of P and find its extreme points. [1]

c) Convert P into the standard equality form. [1]

d) For (c) find all the bases. [2]

e) Find all basic solutions. [6]

f) Find all basic feasible solutions. [1]

g) Find all basic feasible non degenerate solutions. [1]

h) Find all degenerate basic feasible solutions. [1]

Part B

5. Determine the maximum value of $18x_1+4x_2+6x_3$ under the constraints

$3x_1 + x_2 \leq -3$

$2x_1 + x_3 \leq -5$

$x_1 \leq 0, x_2 \leq 0, x_3 \leq 0$

by looking at the dual problem or directly. [4]

6. A factory or firm produces two outputs y and z using a single input x. The set of attainable output levels $H(x)$ from an input use of x is given by $H(x) = \{(y,z) : y^2 + z^2 \leq x\}$. The firm has available to it a maximum of one unit of input x. Let p_1, p_2 denote the price of y, z respectively. Determine the firms optimal output mix, using Kuhn-Tucker theorem. Also find the maximum selling price.

[Hint: Let $f, g_1, g_2, g_3, g_4, g_5 : R^3 \rightarrow R$ be given by

$$\begin{aligned} f(x,y,z) &= p_1 y + p_2 z \\ g_1(x,y,z) &= x - (y^2+z^2) \\ g_2(x,y,z) &= x \\ g_3(x,y,z) &= 1-x \\ g_4(x,y,z) &= y \\ g_5(x,y,z) &= z \end{aligned}$$

$S = \{(x,y,z) : g_i(x,y,z) \geq 0 \text{ for each } i\}$. If (x^*, y^*, z^*) is local maxima for f on S, you can assume for (obvious?) reasons $g_2(x^*, y^*, z^*) > 0, g_4(x^*, y^*, z^*) > 0$ and $g_5(x^*, y^*, z^*) > 0$. [10]

7. a) Let $A = \begin{bmatrix} 5 & 0 \\ 2 & 3 \end{bmatrix}$. Calculate $v_1(A)$. [3]

b) Let $B = \begin{bmatrix} 4 & -1 \\ 1 & 2 \end{bmatrix}$. Calculate $v_2(B)$. [1]

[Hint:

a) <u>*Notation*</u> : For any matrix $A : R^n_{col} \rightarrow R^{m_0}_{col}$ define $v_1(A), v_2(A)$ by $v_1(A) = \max \{\min_i \sum a_{ij}x_j : x_j \geq 0, x_1 + x_2 + \cdots + x_n = 1\}$ and $v_2(A) = \min \{\max_j \sum_i y_i a_{ij} : y_i \geq 0, y_1 + y_2 + \cdots + y_{m_0} = 1\}$.

b) For $A : R^n_{col} \rightarrow R^{m_0}_{col}$, define $B : R^n_{col} \rightarrow R^{m_0}_{col}$ by $b_{ij} = a_{ij} + k$ where k is a fixed real constant. Find a relation between $v_1(B)$ and $v_1(A)$].

8. Let $g, f : R \longrightarrow R$ be given by

$f(x) = x^n \quad n \varepsilon \{2, 3, 4, \cdots\}$,
$g(x) = x$.

Let $D = \{g \geq 0\}$ and $x^* = 0, \lambda = 0$.

a) Show that x^* is not a local maxima for f on D

b) $x^* \varepsilon D, g(x^*) = 0$, rank lin span $\{\bigtriangledown g(x^*)\} = 1, \lambda \geq 0, \lambda g(x^*) = 0$.

c) $(\bigtriangledown f + \lambda \bigtriangledown g)(x^*) = 0$. [3]

9. Let $f : (0, \infty) \times (0, \infty) \to R$ be given by $f(x, y) = x^a y^b, a > 0, b > 0$. If $a + b \leq 1$, then f is concave function. [3]

Full Conversion Table

	Primal	Dual
	$A, \mathbf{x}, \mathbf{b}, \mathbf{c}$	$A^t, \mathbf{y}^t, \mathbf{c}^t, \mathbf{b}^t$
$i \varepsilon I_1,$	$\sum_j a_{ij} x_j = b_i$	$y_i real, y_i \gtreqless 0$
$i \varepsilon I_2,$	$\sum_j a_{ij} x_j \geq b_i$	$y_i \geq 0$
$i \varepsilon I_3,$	$\sum_j a_{ij} x_j \leq b_i$	$y_i \leq 0$
$j \varepsilon J_1,$	$x_j real, x_j \gtreqless 0$	$\sum_i y_i a_{ij} = c_j$
$j \varepsilon J_2,$	$x_j \geq 0$	$\sum_i y_i a_{ij} \leq c_j$
$j \varepsilon J_3,$	$x_j \leq 0$	$\sum_i y_i a_{ij} \geq c_j$
	$min \sum_j c_j x_j$	$max \sum_i y_i b_i$

[1]